# Quasi-Tracklet Fusion Accounting for Cross-Correlation

**Yongxin Gao**
School of Electronic
and Information Engineering
Xi'an Jiaotong University
Xi'an, Shaanxi 710049, P.R. China
yongxingao@gmail.com

**X. Rong Li**
Department of Electrical Engineering
University of New Orleans
New Orleans, LA, U.S.A. 70148
xli@uno.edu

*Abstract*— **This paper deals with distributed fusion with local quasi-tracklets and provides the optimal linear minimum mean-squared error (LMMSE) fusion, namely optimal quasi-tracklet fusion. We analyze its performance, present a necessary and sufficient condition under which the fusion is identical with the centralized fusion, and exploit its relationships with some existing distributed fusion methods. Numerical results are provided to illustrate its performance compared with the centralized and existing distributed fusion algorithms.**

**Keywords: Estimation fusion, distributed fusion, tracklet fusion, linear minimum mean-squared error.**

## I. Introduction

Distributed fusion problems have been the focus of great interest in recent years in multi-sensor environments. This is because centralized fusion (CF), although can provide the globally optimal performance, has heavy computation and data transmission. This naturally leads to distributed fusion and there are numerous results available.

For distributed fusion, there are two commonly used configurations. One is that a global track is maintained at the fusion center and updated when the locally processed data (rather than raw measurements) from local sensors are available, that is, the updated global track is fusion of local data and the historic global track. Normally, fusion in this configuration is done under the LMMSE criterion. The other is that the global track is constructed by a linear combination of local data, and this is generally done under the optimal weighted least squares (OWLS) criterion [1]. Actually, there are some work (see, e.g., [2], [3]) in which the centralized fusion is reconstructed from the local data under some assumptions; however, this is not the focus of this paper.

One characteristic of distributed fusion is whether the locally processed data are state estimates. When the processed data from local sensors are state estimates, the distributed fusion is referred to as the standard distributed fusion; otherwise, it is referred to as the non-standard distributed fusion. For standard distributed fusion, [4], [5] provided the method for computing the error cross-correlations of the local estimates, [6] derived the maximum likelihood (ML) fusion (which is the same as OWLS fusion under the linear Gaussian case) formula, and [7] evaluated the performance of the distributed maximum a posteriori (MAP) fusion (which is the same as LMMSE fusion under the linear Gaussian assumption) with or without feedback. For non-standard distributed fusion, [8], [9] studied the distributed fusion with compressed data, and [10], [11] addressed the problem of fusion with transformed measurements.

The scheme of tracklet fusion was proposed in [12], where a tracklet was computed so that its state estimation errors were not cross-correlated with those of other data in the system. For the system without process noise and under the linear Gaussian assumption, tracklet is equivalent to a state estimate based on only the most recent measurements since the pervious tracklet was sent to the fusion center. That is, the tracklet is formed by estimation just based on the measurements in the *tracklet interval* (the time between two consecutive tracklets sent to the fusion center [12]). As such, local sensors never transmit the information in one measurement more than once to the fusion center, the cross-correlations caused by measurement-errors are avoided and the tracklet fusion achieves the global optimality.

For the system with process noise, the "tracklets", computed similarly to the system without process noise, can not strictly satisfy the original intention of the tracklet concept, since their estimation errors are still cross-correlated and correlated with the error of the global track due to the common process noise. Therefore, using the tracklet method directly to the system with process noise will yield a *quasi-tracklet* or simply *q-tracklet* [12]. And the fusion, which is an approximated method in this case, is referred to as *quasi-tracklet fusion* or simply *q-tracklet fusion*.

*Q-tracklet* fusion is one of the standard distributed fusion methods. For the configuration of restarted global tracks, the cross-covariance of the q-tracklets' estimation errors can be computed by the algorithms of [4], [6], [5] and this will yield the OWLS fusion given local q-tracklets. In this paper, we

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will study the q-tracklet fusion with updated global tracks. The optimal distributed fusion algorithm is derived under the LMMSE criterion and its performance, compared with CF, is also analyzed.

Q-Tracklet fusion can be easily applied to the distributed system in which local sensors are not synchronous (see [12]). In this paper, local sensors are assumed synchronous and the tracklet interval is supposed to have a fixed length $N$. The optimal LMMSE fusion with local q-tracklets (namely, optimal q-tracklets fusion, OQTF) is actually a problem of distributed fusion with reduced-rate communication case. There are also some existing (not necessarily optimal) methods, such as the information matrix fusion (IMF) [2] (see also [13]), the MAP fusion [14] (see also [7]) and the distributed fusion with transformed data [11] (when used with reduced-rate communication), to deal with this problem. We will study the relationships between the OQTF and these existing fusion methods.

The paper is organized as follows. Section II formulates the optimal distributed q-tracklet fusion problem for dynamic systems. Section III gives the distributed LMMSE fusion with local q-tracklets. In section IV, the performance analysis of OQTF is presented. Section V discusses the relationships between OQTF and some existing methods. Numerical results are provided in Section VI. Section VII concludes the paper. Mathematical details are included in the Appendix.

## II. Problem Formulation

The two-sensor (sensor $i$ and $j$) distributed linear dynamic system assumed are given by

$$x_{k+1} = F_k x_k + G_k w_k,\ k = 0, 1, \ldots, \tag{1}$$
$$z_k^l = H_k^l x_k + v_k^l,\ l = i, j \tag{2}$$

where $x_k \in \Re^{n_x}$, $E[x_0] = \bar{x}_0$, and $\text{cov}(x_0) = P_0$. $w$ and $v^l$ are both zero-mean white noises. It is also assumed that measurement noises are uncorrelated across sensors, that is, $\text{cov}(v_k^i, v_k^j) = 0$ and $w$, $v^l$ and $x_0$ are uncorrelated with each other. The covariances of the noises $w_k$ and $v_k^l$ are given by

$$\text{cov}(w_k) = Q_k,\ \text{cov}(v_k^l) = R_k^l$$

where $R_k^l$ are invertible for all $l$ and $k$.

Given the tracklet interval $N$, we have the pseudo-measurement equation

$$Z_{k+1:k+N}^l = \tilde{H}_{k+N}^l x_{k+N} + \tilde{V}_{k+N}^l,\ l = i, j$$

where

$$Z_{k+1:k+N}^l = \left[ (z_{k+N}^l)', (z_{k+N-1}^l)', \cdots, (z_{k+1}^l)' \right]'$$

$$\tilde{H}_{k+N}^l = \begin{bmatrix} H_{k+N}^l \\ H_{k+N-1}^l (F_{k+N-1})^{-1} \\ \vdots \\ H_{k+1}^l \prod_{j=1}^{N-1} (F_{k+j})^{-1} \end{bmatrix}$$

$\tilde{V}_{k+N}^l$ is the corresponding pseudo-measurement error and

$$\tilde{R}_{k+N}^l = \text{cov}(\tilde{V}_{k+N}^l),\ \tilde{R}_{k+N}^{i,j} = \text{cov}(\tilde{V}_{k+N}^i, \tilde{V}_{k+N}^j)$$

Therefore, the local q-tracklet pertinent to sensor $l$ at time $k+N$ is

$$\hat{x}_{k+N}^l = P_{k+N}^l (\tilde{H}_{k+N}^l)' (\tilde{R}_{k+N}^l)^{-1} Z_{k+1:k+N}^l \tag{3}$$
$$P_{k+N}^l = [(\tilde{H}_{k+N}^l)' (\tilde{R}_{k+N}^l)^{-1} \tilde{H}_{k+N}^l]^{-1} \tag{4}$$

Note that here $\tilde{H}_{k+N}^l$ for all $l$ should be of full column rank. Otherwise, the q-tracklet $\hat{x}_{k+N}^l$ will not exist. It is obvious that $\hat{x}_{k+N}^l$ is obtained by least square (LS) method based on the measurements in the tracklet interval. When $Q = 0$, estimation errors of $\hat{x}_{k+N}^i$ and $\hat{x}_{k+N}^j$ are uncorrelated with each other and with the error of the global track and thus $\hat{x}_{k+N}^l$ is called a tracklet; when $Q \neq 0$, the estimation errors are correlated and $\hat{x}_{k+N}^l$ is called a q-tracklet.

Optimal fusion with q-tracklets should consider these error cross-correlations. For the configuration with updated global tracks, the problem is to optimally estimate $x_{k+N}$ based on the local q-tracklets $\hat{x}_{k+N}^l$, $l = i, j$, and the previous global track $\hat{x}_k^d$, i.e.,

$$\hat{x}_{k+N}^d = E^+[x_{k+N} | \hat{x}_{k+N}^i, \hat{x}_{k+N}^j, \hat{x}_k^d] \tag{5}$$
$$P_{k+N}^d = \text{MSE}(\hat{x}_{k+N}^d) \tag{6}$$

where $E^+[\cdot]$ denotes the best linear unbiased estimation (BLUE) without knowing the prior [1], [15]. Unlike the normal standard distributed estimation fusion, where $\hat{x}_{k+N}^l$ is an estimate based on all historic measurements of sensor $l$, here $\hat{x}_{k+N}^l$ is just based on the measurements in the tracklet interval.

Extrapolate $(\hat{x}_k^d, P_k^d)$ to time $k+N$ to get $(\hat{x}_{k+N|k}^d, P_{k+N|k}^d)$

$$\hat{x}_{k+N|k}^d = \prod_{n=0}^{N-1} F_{k+n} \hat{x}_k^d$$
$$P_{k+N|k}^d = F_{k+N-1} P_{k+N-1|k}^d F'_{k+N-1} + G_{k+N-1} Q_{k+N-1} G'_{k+n-1}$$

where $P_{k|k}^d = P_k^d$. We can express $\hat{x}_{k+N|k}^d$ as follows

$$\hat{x}_{k+N|k}^d = I_{n_x} x_{k+N} + u_{k+N}$$

where $I_{n_x}$ is the $n_x$-dimensional identity matrix, $u_{k+N}$ denotes a pseudo-measurement noise, $E[u_{k+N}] = 0$ and $\text{cov}(u_{k+N}) = P_{k+N|k}^d$. Note that the estimate $\hat{x}_{k+N}^d$ always exists, because $\hat{x}_{k+N}^i$, $\hat{x}_{k+N}^j$ and $\hat{x}_{k+N|k}^d$ are unbiased estimates of $x_{k+N}$ themselves. Since $I_{n_x}$ is invertible, by the Theorem 4.2 of [15], we can convert Eq. (5) into an LMMSE estimation without any information loss, i.e.,

$$\hat{x}_{k+N}^d = E^*[x_{k+N} | \hat{x}_{k+N}^i, \hat{x}_{k+N}^j] \tag{7}$$

where $E^*[\cdot]$ denotes the LMMSE estimator with prior and the prior information of $x_{k+N}$ is $(\hat{x}_{k+N|k}^d, P_{k+N|k}^d)$.

## III. Optimal LMMSE Fusion with Q-tracklets

View the local q-tracklet $\hat{x}_{k+N}^l$ as an observation of the estimand (i.e., what is to be estimated) $x_{k+N}$ by the following identity [1]:

$$\hat{x}_{k+N}^l = x_{k+N} + (\hat{x}_{k+N}^l - x_{k+N})$$

The fusion problem (7) can be solved as follows [1], [16],

$$K^d_{k+N} = (P^d_{k+N|k}\tilde{I}' + C^d_{k+N})(S^d_{k+N})^{-1} \tag{8}$$

$$\begin{aligned} S^d_{k+N} &= \tilde{I}P^d_{k+N|k}\tilde{I}' + R^d_{k+N} \\ &\quad + \tilde{I}C^d_{k+N} + (C^d_{k+N})'\tilde{I}' \end{aligned} \tag{9}$$

$$\hat{x}^d_{k+N} = \hat{x}^d_{k+N|k} + K^d_{k+N}(Y_{k+N} - \tilde{I}\hat{x}^d_{k+N|k}) \tag{10}$$

$$P^d_{k+N} = P^d_{k+N|k} - K^d_{k+N}S^d_{k+N}(K^d_{k+N})' \tag{11}$$

where

$$\begin{aligned} Y_{k+N} &= [(\hat{x}^i_{k+N})', (\hat{x}^j_{k+N})']' \\ \tilde{I} &= \begin{bmatrix} I_{n_x} & I_{n_x} \end{bmatrix}' \\ C^d_{k+N} &= \begin{bmatrix} C^{*i}_{k+N} & C^{*j}_{k+N} \end{bmatrix} \\ R^d_{k+N} &= \begin{bmatrix} P^i_{k+N} & P^{i,j}_{k+N} \\ P^{j,i}_{k+N} & P^j_{k+N} \end{bmatrix} \end{aligned}$$

The cross-covariance of the local q-tracklets estimation errors $P^{i,j}_{k+N}$ can be computed by the method of [4], i.e.,

$$\begin{aligned} P^{i,j}_{m+1} &= (I_{n_x} - K^i_m H^i_m) \\ &\quad \cdot (F_m P^{i,j}_m F'_m + G_m Q_m G'_m)(I_{n_x} - K^j_m H^j_m)' \end{aligned} \tag{12}$$

where $K^l_m$ is the filter gain pertinent to sensor $l$ at time $m$. Here $m = k+1, \ldots, k+N-1$, $P^{i,j}_{k+1} = \mathbf{0}_{n_x}$ and $P^{j,i}_{m+1}$ follows from Eq. (12) with $i$ and $j$ interchanged.

The error-covariance between the global track and local q-tracklet $C^{*i}_{k+N}$ can be computed as follows (see the Appendix)

$$\begin{aligned} C^{*i}_{m+1} &= (F_m C^{*i}_m F'_m - G_m Q_m G'_m) \\ &\quad \cdot (I_{n_x} - K^i_{m+1} H^i_{m+1})' \end{aligned} \tag{13}$$

where $m = k+1, \ldots, k+N-1$, $C^{*i}_{k+1} = \mathbf{0}_{n_x}$ and $C^{*j}_{m+1}$ follows from Eq. (13) by replacing $i$ by $j$.

*Remark:* The local q-tracklets are obtained by filtering with no initial prior (see Eq. (3)), and thus the information filter is more suitable here than the Kalman filter [17]. Therefore, it is better to compute $P^{i,j}_{m+1}$ and $C^{*i}_{m+1}$ based on the quantities of information filter rather than the Kalman filter. So, the following recursion is preferred to Eq. (12) (see the Appendix):

$$P^{i,j}_{k+N} = P^i_{k+N} D^{ij}_{k+N} P^j_{k+N} \tag{14}$$

$D^{ij}_{k+N}$ is the corresponding error-covariance of the information state which can be recursively computed as follows

$$\begin{aligned} D^{ij}_{m+1} &= U^i_m(D^{ij}_m \\ &\quad + I^i_m F^{-1}_m G_m Q_m G'_m (F^{-1}_m)' I^j_m)(U^j_m)' \\ U^l_m &= (I_{n_x} - A^l_m T^l_m)(F^{-1}_m)',\ l = i, j \\ m &= k+1, \ldots, k+N-1 \end{aligned} \tag{15}$$

where (see [17])

$$\begin{aligned} A^l_m &= (F^{-1}_m)' I^l_m F^{-1}_m \\ T^l_m &= G_m(Q^{-1}_m + G'_m A_m G_m)^{-1} G'_m \end{aligned}$$

and $I^l_m$ is the information matrix at time $m$.

The following recursion is preferred to Eq. (13) (see the Appendix):

$$C^{*l}_{k+N} = D^{*l}_{k+N} P^l_{k+N} \tag{16}$$

and $D^{*l}_{k+N}$ can be obtained recursively by

$$D^{*l}_{m+1} = (F_m D^{*l}_m - G_m Q_m G'_m (F^{-1}_m)' I^l_m)(U^l_m)' \tag{17}$$

The above recursions are started with $D^{ij}_{k+1} = \mathbf{0}_{n_x}$, $D^{*l}_{k+1} = \mathbf{0}_{n_x}$ and $I^l_{k+1} = (H^l_{k+1})'(R^l_{k+1})^{-1}H^l_{k+1}$.

## IV. Performance Analysis

Clearly, the above fusion, although optimal in the LMMSE sense, is based on the q-tracklets rather than raw measurements and thus, as is well known, can not outperform CF. The next problem is under what condition the optimal LMMSE fusion with q-tracklets will have the same performance as CF. In this section, particularly in Theorem 1, we will deal with this problem.

Let

$$\begin{aligned} Z^c_{k+N} &= [(Z^i_{k+1:k+N})', (Z^j_{k+1:k+N})']' \\ H^c_{k+N} &= [(\tilde{H}^i_{k+N})', (\tilde{H}^j_{k+N})']' \\ V^c_{k+N} &= [(\tilde{V}^i_{k+N})', (\tilde{V}^j_{k+N})']' \end{aligned}$$

Assume that the CF result at time $k$ is $(\hat{x}^c_k, P^c_k)$ and this result propagates to time $k+N$ to be viewed as the prediction of $x_{k+N}$. Then the optimal LMMSE CF can be computed as follows [1], [16],

$$\hat{x}^c_{k+N|k} = \prod_{n=0}^{N-1} F_{k+n}\hat{x}^c_k \tag{18}$$

$$\begin{aligned} P^c_{k+N|k} &= F_{k+N-1}P^c_{k+N-1|k}F'_{k+N-1} \\ &\quad + G_{k+N-1}Q_{k+N-1}G'_{k+N-1} \end{aligned} \tag{19}$$

$$K^c_{k+N} = (P^c_{k+N|k}(H^c_{k+N})' + C^c_{k+N})(S^c_{k+N})^{-1} \tag{20}$$

$$\begin{aligned} S^c_{k+N} &= H^c_{k+N}P^c_{k+N|k}(H^c_{k+N})' + R^c_{k+N} \\ &\quad + H^c_{k+N}C^c_{k+N} + (H^c_{k+N}C^c_{k+N})' \end{aligned} \tag{21}$$

$$\hat{x}^c_{k+N} = \hat{x}^c_{k+N|k} + K^c_{k+N}(Z^c_{k+N} - H^c_{k+N}\hat{x}^c_{k+N|k}) \tag{22}$$

$$P^c_{k+N} = P^c_{k+N|k} - K^c_{k+N}S^c_{k+N}(K^c_{k+N})' \tag{23}$$

where

$$\begin{aligned} C^c_{k+N} &= \begin{bmatrix} C^{*i}_{x_{k+N}V} & C^{*j}_{x_{k+N}V} \end{bmatrix} \\ R^c_{k+N} &= \begin{bmatrix} \tilde{R}^i_{k+N} & \tilde{R}^{i,j}_{k+N} \\ \tilde{R}^{j,i}_{k+N} & \tilde{R}^j_{k+N} \end{bmatrix} \end{aligned}$$

$P^c_{k|k} = P^c_k$ and

$$C^{*l}_{x_{k+N}V} = E[(x_{k+N} - \hat{x}^c_{k+N|k})(\tilde{V}^l_{k+N})']$$

Let

$$D = H^c_{k+N}P^c_{k+N|k} + (C^c_{k+N})' \tag{24}$$

$$U = \mathrm{diag}[(\tilde{H}^i_{k+N})'(\tilde{R}^i_{k+N})^{-1}, (\tilde{H}^j_{k+N})'(\tilde{R}^j_{k+N})^{-1}] \tag{25}$$

$$W = \mathrm{diag}(P^i_{k+N}, P^j_{k+N}) \tag{26}$$

For the assumed system, we have the following theorem to show the performance of OQTF.

*Theorem 1:* Given that $P_k^d = P_k^c$, then the OQTF is identical with CF at time $k+N$ (i.e., $P_{k+N}^d = P_{k+N}^c$), if and only if

$$U'(U')^+(S_{k+N}^c)^{-1}D = (S_{k+N}^c)^{-1}D \tag{27}$$

where $A^+$ stands for the Moore-Penrose pseudo-inverse (MP inverse in short) of $A$.

*Proof:* From Eqs. (3), (11), (23), (25) and (26), we have

$$\begin{aligned} & K_{k+N}^d S_{k+N}^d (K_{k+N}^d)' \\ & = (P_{k+N|k}^d \tilde{I}' + C_{k+N}^d)(S_{k+N}^d)^{-1}(P_{k+N|k}^d \tilde{I}' + C_{k+N}^d)' \\ & = D'U'W(WUS_{k+N}^c U'W)^{-1}WUD \qquad (28) \\ & = D'U'(US_{k+N}^c U')^{-1}UD \qquad (29) \end{aligned}$$

where

$$P_{k+N|k}^d = P_{k+N|k}^c$$

due to

$$P_k^d = P_k^c$$

It is well known that

$$P_{k+N}^c \leq P_{k+N}^d$$

Then it follows from Eqs. (11) and (23) that

$$K_{k+N}^d S_{k+N}^d (K_{k+N}^d)' \leq K_{k+N}^c S_{k+N}^c (K_{k+N}^c)'$$

Thus, we have

$$D'U'(US_{k+N}^c U')^{-1}UD \leq D'(S_{k+N}^c)^{-1}D$$

Therefore, for any $d \in$ Range($D$), the following inequality holds

$$d'U'(US_{k+N}^c U')^{-1}Ud \leq d'(S_{k+N}^c)^{-1}d$$

Define a new inner product of $d \in$ Range($D$) [10] and

$$\langle d, d\rangle = d'(S_{k+N}^c)^{-1}d$$

It follows that

$$\begin{aligned} \langle d, d\rangle & = d'(S_{k+N}^c)^{-1}d \\ & \geq d'U'(US_{k+N}^c U')^{-1}Ud \\ & = \langle d, S_{k+N}^c U'(US_{k+N}^c U')^{-1}Ud\rangle \end{aligned}$$

Because $S_{k+N}^c U'(US_{k+N}^c U')^+ U$ is a projection, the equality holds if and only if

$$\text{Range}(D) \subset \text{Range}(S_{k+N}^c U'(US_{k+N}^c U')^{-1}U)$$

Given that Range($D$) $\subset$ Range($S_{k+N}^c U'(US_{k+N}^c U')^{-1}U$), there must exist an $X$ such that

$$S_{k+N}^c U'(US_{k+N}^c U')^{-1}UX = D \tag{30}$$

which is equivalent to

$$U'\bar{X} = (S_{k+N}^c)^{-1}D \tag{31}$$

Thus, a necessary and sufficient condition [18] for the above equation to have a solution is

$$U'(U')^+(S_{k+N}^c)^{-1}D = (S_{k+N}^c)^{-1}D$$

This completes the proof. ∎

*Remark 1:* Clearly, it is the common process noise that makes the errors of local q-tracklets cross-correlated and correlated with the error of the global track. When the process noise is absent, the above distributed fusion reduces to the tracklet fusion (with the updated global track) and thus is globally optimal [12]. This can also be proved from Eq. (27) as follows:

$$\begin{aligned} & U'(U')^+(S_{k+N}^c)^{-1}D \\ & = U'(U')^+[H_{k+N}^c P_{k+N|k}^c (H_{k+N}^c)' + R_{k+N}^c]^{-1} \\ & \quad \cdot H_{k+N}^c P_{k+N|k}^c \\ & = U'(U')^+[(R_{k+N}^c)^{-1} - (R_{k+N}^c)^{-1}H_{k+N}^c P_{k+N|k}^c \\ & \quad \cdot (H_{k+N}^c)'(R_{k+N}^c)^{-1}]H_{k+N}^c P_{k+N|k}^c \\ & = U'(U')^+[U'\tilde{I}P_{k+N|k}^c - U'\tilde{I}P_{k+N|k}^c \tilde{I}'UH_{k+N}^c P_{k+N|k}^c] \\ & = U'\tilde{I}P_{k+N|k}^c - U'\tilde{I}P_{k+N|k}^c \tilde{I}'UH_{k+N}^c P_{k+N|k}^c \\ & = (R_{k+N}^c)^{-1}H_{k+N}^c P_{k+N|k}^c - (R_{k+N}^c)^{-1}H_{k+N}^c P_{k+N|k}^c \\ & \quad \cdot (H_{k+N}^c)'(R_{k+N}^c)^{-1}H_{k+N}^c P_{k+N|k}^c \\ & = [(R_{k+N}^c)^{-1} - (R_{k+N}^c)^{-1}H_{k+N}^c P_{k+N|k}^c \\ & \quad \cdot (H_{k+N}^c)'(R_{k+N}^c)^{-1}]H_{k+N}^c P_{k+N|k}^c \\ & = (S_{k+N}^c)^{-1}D \end{aligned}$$

where

$$(R_{k+N}^c)^{-1}H_{k+N}^c = U'\tilde{I}$$

due to the uncorrelatedness of $\tilde{V}_{k+N}^i$ and $\tilde{V}_{k+N}^j$.

*Remark 2:* Another apparent condition for the above distributed fusion to achieve the globally optimal performance is that $\tilde{H}_{k+N}^l$ for all $l$ are of full row rank. Since $\tilde{H}_{k+N}^l$ also should be of full column rank, then $\tilde{H}_{k+N}^l$ is invertible. In this case,

$$U'(U')^+ = U'(U')^{-1} = I$$

and Eq. (27) is satisfied.

## V. Relationships and Comparisons

In OQTF, local sensors need to transmit their q-tracklets to the fusion center every $N$ time instants. Thus OQTF is actually a distributed fusion scheme with reduced-rate communication. There are also some existing methods which can also address this problem. We discuss their performance and relationships with OQTF in this section.

### A. Optimal Distributed Fusion with Transformed Data

An optimal distributed fusion rule with transformed data (FWTD) was proposed in [11], where local sensors transmit the transformed data $(H_k^l)'(R_k^l)^{-1}z_k^l$ for all $l$ to the fusion center every sampling time and the fusion center does fusion under the LMMSE criterion. When this scheme is used in the reduced-rate communication case, where local sensors transmit $(\tilde{H}_{k+N}^l)'(\tilde{R}_{k+N}^l)^{-1}Z_{k+1:k+N}^l$ to the fusion center and the optimal fusion is also done in the LMMSE sense, it will have the same performance as OQTF, as shown next.

FWTD has the mean-squared error (MSE) matrix [11]

$$\begin{aligned}P^t_{k+N} &= P^t_{k+N|k} - (P^t_{k+N|k}(H^c_{k+N})' + C^c_{k+N})\\ &\quad U'(US^c_{k+N}U')^{-1}U(H^c_{k+N}P^t_{k+N|k} + (C^c_{k+N})')\end{aligned}$$

where the superscript $t$ denotes the quantities obtained by the method of FWTD. Therefore, by Eqs. (28)-(29), we can see that if $P^t_{k+N|k} = P^d_{k+N|k}$ then

$$P^t_{k+N} = P^d_{k+N}$$

Thus, FWTD and OQTF are identical (with probability 1).

### B. Information Matrix Fusion

When IMF is used with reduced-rate communication, it has [13]

$$\begin{aligned}(P^m_{k+N})^{-1}\hat{x}^m_{k+N} &= (P^m_{k+N|k})^{-1}\hat{x}^m_{k+N|k}\\ &\quad + \sum_l [(P^l_{k+N|k+N})^{-1}\hat{x}^l_{k+N|k+N} - (P^l_{k+N|k})^{-1}\hat{x}^l_{k+N|k}]\\ (P^m_{k+N})^{-1} &= (P^m_{k+N|k})^{-1}\\ &\quad + \sum_l [(P^l_{k+N|k+N})^{-1} - (P^l_{k+N|k})^{-1}]\end{aligned}$$

where the superscript $m$ denotes quantities obtained by IMF.

Obviously, when the process noise is absent ($Q = 0$), we have

$$\begin{aligned}&(P^l_{k+N|k+N})^{-1}\hat{x}^l_{k+N|k+N} - (P^l_{k+N|k})^{-1}\hat{x}^l_{k+N|k}\\ &= (\tilde{H}^l_{k+N})'(\tilde{R}^l_{k+N})^{-1}Z^l_{k+1:k+N}\end{aligned}$$

which means that IMF is identical with the FWTD in this case. Thus, when there is no process noise, the IMF, FWTD and OQTF fusion are identical, and they all have the globally optimal performance. In the other extreme that $Q$ is very large, the state estimate $\hat{x}_k$ is primarily based on the current measurements $z^l_k$, since the system model is unreliable and historic measurements have little contribution to the (current) estimation. Thus, for CF and IMF, we have

$$\begin{aligned}\hat{x}^c_k &\approx E^+[x_k|z^i_k, z^j_k]\\ &\approx (\sum_l (P^l_{k+N|k+N})^{-1}) \sum_l (P^l_{k+N|k+N})^{-1}\hat{x}^l_{k+N|k+N}\\ &\approx \hat{x}^m_{k+N}\end{aligned}$$

because $(P^m_{k+N|k})^{-1}$ and $(P^l_{k+N|k})^{-1}$ approach zero matrices. It means that when $Q$ is very large, IMF and CF will have the comparable performance. The same conclusion can be made for OQTF and CF.

As discussed above, for a very small or large $Q$, the performance of OQTF, IMF and CF are similar. But for a regular (not too small or large) $Q$, IMF and OQTF have their pros and cons:

*1)* IMF is not credible: $P^m_{k+N}$ is not really close to the actual MSE matrix of $\hat{x}^m_{k+N}$. OQTF is credible [19], since it is an optimal method and a truly optimal estimator is always credible [17] (termed consistency there).

*2)* IMF is simple and easy to implement. OQTF is relatively complex and needs more computation.

### C. MAP Fusion with Feedback

MAP fusion is identical with the LMMSE fusion under the linear Gaussian assumption [7]. For the assumed system (1)-(2), if we initialize the local Kalman filters with the fusion result of the previous step and let

$$C^{*l}_{k+1} = (-F_kP^d_kF'_m - G_kQ_kG'_k)(I_{n_x} - K^l_{k+1}H^l_{k+1})' \quad (32)$$

$$\begin{aligned}P^{i,j}_{k+1} &= (I_{n_x} - K^i_kH^i_k)\\ &\quad \cdot (F_kP^d_kF'_k + G_kQ_kG'_k)(I_{n_x} - K^j_kH^j_k)'\end{aligned} \quad (33)$$

then it turns out that the algorithms (8)–(11) are the MAP fusion with feedback (MAPwF).

When $N$ is large, the reinitializtion will have little effect on the fusion, that is,

$$\begin{aligned}\hat{x}^l_{k+N} &= E^+[x_{k+N}|Z^l_{k+1:k+N}]\\ &\approx E^+[x_{k+N}|\hat{x}^d_k, Z^l_{k+1:k+N}]\end{aligned}$$

This means that when a large set of measurements is used, the estimation with or without prior differs slightly. Thus, when $N$ is large, the MAPwF and OQTF have comparable performance.

## VI. Numerical Examples

In this section, we provide several illustrative examples to verify our proposed fusion algorithms. All these examples are for the following kinematic model [6] of a target tracked by two sensors:

$$x_{k+1} = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} x_k + \begin{bmatrix} \frac{T^2}{2} \\ T \end{bmatrix} w_k \quad (34)$$

$$z^l_k = \begin{bmatrix} 1 & 0 \end{bmatrix} x_k + v^l_k,\ l = 1, 2 \quad (35)$$

where sampling time $T = 1$, $w_k$ is zero mean white Gaussian process noise with variance $q$, and the two measurement noises are Gaussian and mutually independent with zero mean and variance $R^l_k = 20$.

The two synchronous measurements have a fixed rate and the same measurement model, and each generates a 1-dimensional measurement. 500 Monte Carlo runs are conducted over a total time span of 120 seconds.

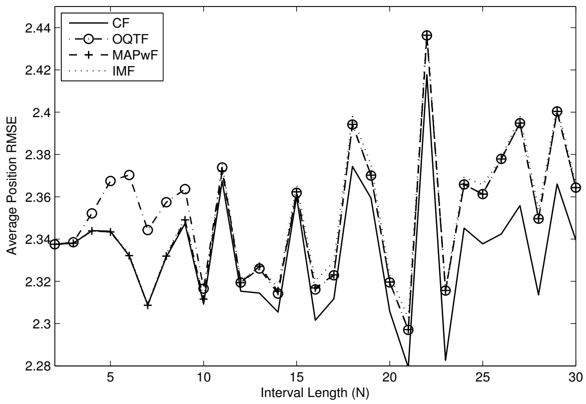
The true initial state is generated Gaussian distributed with mean and covariance:

$$\bar{x}_0 = \begin{bmatrix} 100 \\ 1 \end{bmatrix},\ P_0 = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$$

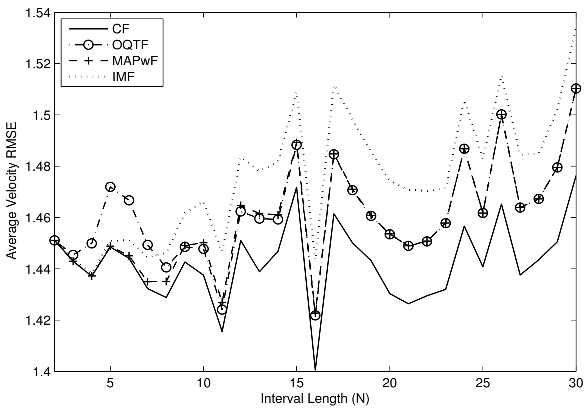
which are also used to initialize the centralized filter. Information filters are employed in local sensors and initialized with zero information.

For each example, the root mean square errors (RMSE) or average RMSE over the total time span are used to illustrate the performance of the fusion algorithms and the noncredibility index (NCI) [19] is used to measure their credibility.

Fig. 1 shows the average position and velocity RMSE of CF, OQTF, MAPwF and IMF vs. interval length ($N$) for $q = 1$. As can be seen, when $N > 10$, OQTF and MAPwF have almost the same performance in position estimation; when $10 < N < 16$, OQTF outperforms MAPwF slightly in velocity

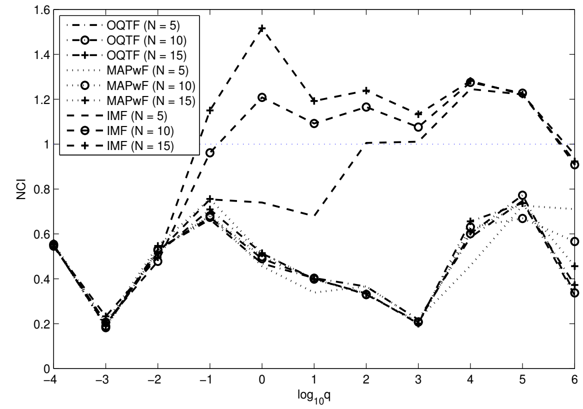



(a) Average Position RMSE of CF, OQTF, MAPwF and IMF

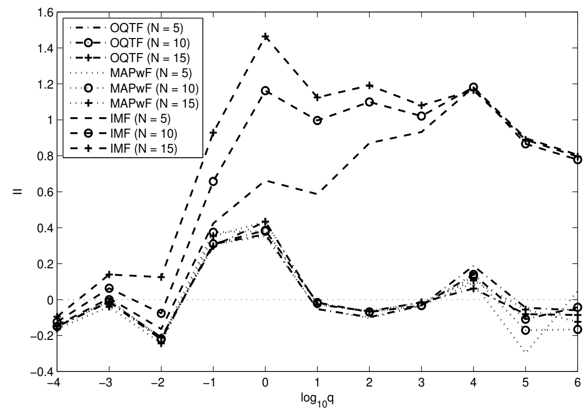



(b) Average Velocity RMSE of CF, OQTF, MAPwF and IMF

Fig. 1. Average position and velocity RMSE vs. interval length (N).




(a) NCI of CF, OQTF, MAPwF and IMF




(b) II of CF, OQTF, MAPwF and IMF

Fig. 2. NCI and II vs. covariance q.

estimation. When $N < 7$, IMF is more accurate than OQTF. As $N$ increases, IMF performs worst.

Fig. 2 shows the NCI and inclination indicator (II) values [19] of OQTF, MAPwF and IMF at time $k = 120$ vs. the variance $q$ of process noise. As can be seen, OQTF and MAPwF are credible, since their NCI values are smaller than 1. IMF is credible when $q$ is small (e.g., $q < 10^{-2}$). When $q$ is large, IMF is optimistic since its NCI values are larger than 1 and close to the II values, which are positive.

## VII. Conclusions

Theoretically, the OQTF is similar to the standard distributed estimation fusion with reduced-rate communication, except that the local estimates are just based on the measurements in the tracklet interval. In this paper, we have presented the q-tracklet fusion accounting for cross-correlation, i.e., the optimal LMMSE q-tracklet fusion, and also analyzed its performance. A necessary and sufficient condition under which the OQTF is identical with CF has been derived. We have compared the performance of OQTF with CF and existing methods via numerical simulations.

## Appendix

### A. Derivation of Eq. (13)

$$\begin{aligned}
C^{*i}_{m+1} &= E[(x_{m+1} - \hat{x}^d_{m+1|k})(\hat{x}^i_{m+1} - x_{m+1})'] \\
&= E[(F_m x_m + G_m w_m - F_m \hat{x}^d_{m|k}) \\
&\cdot ((I_{n_x} - K^i_{m+1} H^i_{m+1})\hat{x}^i_{m+1|m} \\
&+ K^i_{m+1} z^i_{m+1} - x_{m+1})'] \\
&= E[(F_m(x_m - \hat{x}^d_{m|k}) + G_m w_m) \\
&\cdot ((I_{n_x} - K^i_{m+1} H^i_{m+1})(\hat{x}^i_{m+1|m} - x_{m+1}))'] \\
&= E[(F_m(x_m - \hat{x}^d_{m|k}) + G_m w_m) \\
&\cdot ((I_{n_x} - K^i_{m+1} H^i_{m+1})(F_m(\hat{x}^i_m - x_m) - G_m w_m))'] \\
&= (F_m C^{*i}_m F'_m - G_m Q_m G'_m)(I_{n_x} - K^i_{m+1} H^i_{m+1})'
\end{aligned}$$

### B. Derivation of Eqs. (14) and (15)

Let

$$\begin{aligned}
D^{ij}_{m+1} = E[((P^i_{m+1})^{-1}\hat{x}^i_{m+1} - (P^i_{m+1})^{-1}x_{m+1}) \\
\cdot ((P^j_{m+1})^{-1}\hat{x}^j_{m+1} - (P^j_{m+1})^{-1}x_{m+1})']
\end{aligned}$$

Then

$$
\begin{aligned}
P^{i,j}_{k+N} &= E[(\hat{x}^i_{k+N} - x_{k+N})(\hat{x}^j_{k+N} - x_{k+N})'] \\
&= P^i_{k+N} E[((P^i_{k+N})^{-1}\hat{x}^i_{k+N} - (P^i_{k+N})^{-1}x_{k+N}) \\
&\quad \cdot ((P^j_{k+N})^{-1}\hat{x}^j_{k+N} - (P^j_{k+N})^{-1}x_{k+N})']P^j_{k+N} \\
&= P^i_{k+N} D^{ij}_{k+N} P^j_{k+N}
\end{aligned}
$$

Since

$$
\begin{aligned}
&(P^i_{m+1})^{-1}\hat{x}^i_{m+1} - (P^i_{m+1})^{-1}x_{m+1} \\
&= \hat{y}^i_{m+1} - I^i_{m+1}x_{m+1} \\
&= \hat{y}^i_{m+1|m} + (H^i_{m+1})'(R^i_{m+1})^{-1}z^i_{m+1} - I^i_{m+1}x_{m+1} \\
&= (I_{n_x} - A^i_m T^i_m)(F^{-1}_m)'\hat{y}^i_m \\
&+ ((H^i_{m+1})'(R^i_{m+1})^{-1}H^i_{m+1} - I_{m+1})x_{m+1} \\
&+ (H^i_{m+1})'(R^i_{m+1})^{-1}v^i_{m+1} \\
&= (I_{n_x} - A^i_m T^i_m)(F^{-1}_m)'\hat{y}^i_m \\
&- (I_{n_x} - A^i_m T^i_m)(F^{-1}_m)'I^i_m(x_m + F^{-1}_m G_m w_m) \\
&+ (H^i_{m+1})'(R^i_{m+1})^{-1}v^i_{m+1} \\
&= (I_{n_x} - A^i_m T^i_m)(F^{-1}_m)' \\
&\cdot (\hat{y}^i_{m|m} - I^i_m x_m - I^i_m F^{-1}_m G_m w_m) \\
&+ (H^i_{m+1})'(R^i_{m+1})^{-1}v^i_{m+1}
\end{aligned} \tag{36}
$$

where $\hat{y}^i_{m+1|m}$ is the one-step prediction of $\hat{y}^i_m$, we have

$$
\begin{aligned}
D^{ij}_{m+1} &= (I_{n_x} - A^i_m T^i_m)(F^{-1}_m)' \\
&\cdot (D^{ij}_m + I^i_m F^{-1}_m G_m Q_m G'_m (F^{-1}_m)' I^j_m) \\
&\cdot F^{-1}_m (I_{n_x} - A^j_m T^j_m)'
\end{aligned}
$$

### *C. Derivation of Eqs. (16) and (17)*

Let

$$
\begin{aligned}
D^{*l}_{k+N} &= E[(x_{k+N} - \hat{x}^d_{k+N|k}) \\
&\cdot ((P^l_{k+N})^{-1}\hat{x}^l_{k+N} - (P^l_{k+N})^{-1}x_{k+N})']
\end{aligned}
$$

Then

$$
\begin{aligned}
C^{*l}_{k+N} &= E[(x_{k+N} - \hat{x}^d_{k+N|k})(\hat{x}^l_{k+N} - x_{k+N})'] \\
&= E[(x_{k+N} - \hat{x}^d_{k+N|k}) \\
&\cdot ((P^l_{k+N})^{-1}\hat{x}^l_{k+N} - (P^l_{k+N})^{-1}x_{k+N})']P^l_{k+N} \\
&= D^{*l}_{k+N} P^l_{k+N}
\end{aligned}
$$

Since

$$
\begin{aligned}
x_{m+1} - \hat{x}^d_{m+1|k} &= F_m x_m + G_m w_m - F_m \hat{x}^d_{m|k} \\
&= F_m(x_m - \hat{x}^d_{m|k}) + G_m w_m
\end{aligned}
$$

and following Eq. (36), we have

$$
\begin{aligned}
C^{*l}_{m+1} &= [F_m C^{*l}_m - G_m Q_m G'_m (F^{-1}_m)' I^l_m] \\
&\cdot F^{-1}_m (I_{n_x} - A^l_m T^l_m)'
\end{aligned}
$$

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